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Unified Treatment of the Luminosity Distance in Cosmology

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Abstract

Comparing the luminosity distance measurements to its theoretical predictions is one of the cornerstones in establishing the modern cosmology. However, as shown in Biern & Yoo, its theoretical predictions in literature are often plagued with infrared divergences and gauge-dependences. This trend calls into question the sanity of the methods used to derive the luminosity distance. Here we critically investigate four different methods — the geometric approach, the Sachs approach, the Jacobi mapping approach, and the geodesic light cone (GLC) approach to modeling the luminosity distance, and we present a unified treatment of such methods, facilitating the comparison among the methods and checking their sanity. All of these four methods, if exercised properly, can be used to reproduce the correct description of the luminosity distance.

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Contents

1 Introduction

Measurements of the luminosity distance from distant supernovas provided one of the most important evidence for the mysterious substance in the Universe, or dark energy [4, 5]. However, it has been well known that the standard procedure in supernova cosmology ignores the effect of inhomogeneities in the Universe, in comparing the observations to the theoretical predictions of the luminosity distance. Accordingly, there have been numerous theoretical work to account for the inhomogeneities in the Universe in predicting the luminosity distance.

Within the cosmological framework, the luminosity distance in an inhomogeneous universe was first computed in its complete form [1] by using the optical scalar equation, and an explicit check of gauge invariance was also made in [1]. Since this pioneering work, other methods such as the Jacobi mapping [3], the geometric approach [6–8], the geodesic light cone (GLC) approach [9] were employed in computing the luminosity distance, and the calculations were already extended to the second order in perturbations, using the optical scalar equation [2, 10, 11], the geometric approach [8] and the GLC approach [12–16]. Such second-order calculations are needed to compute the leading-order corrections to the mean of the luminosity distance, in addition to computing its variance. However, these second-order calculations are complicated by nature, and it is often difficult to compare the results of calculations by two groups even with the identical approach adopted. Furthermore, Biern & Yoo [17] showed that most of the luminosity distance calculations in literature, even at the linear order in perturbations, are often plagued with infrared divergences, as the gauge-invariance of its expression is broken in those calculations.

To the linear order in perturbations, at least, this situation makes little sense, as the full gauge-invariance of the luminosity distance calculation was explicitly verified [1, 6, 7, 18, 19] under a general coordinate transformation. However, when numerical calculations are performed in practice, a certain gauge condition is adopted to facilitate the computation, and several terms through the process are ignored in the calculations, breaking the gauge-invariance of the expression. Consequently, some sick features in theory arise, such as the infrared divergences of the variance in the luminosity distance [10]. Often this pathology is avoided in most numerical calculations by introducing a cut-off scale with little justification. However, as demonstrated in [17], the correct gauge-invariant calculation of the luminosity distance is devoid of such pathology, and other numerical errors in the luminosity distance can be avoided by ensuring the gauge-invariance of its expression.

In comparison to the Planck result [20], recent measurements [21] of the luminosity distance and hence the Hubble parameter show signs of discrepancies, and this conflict has renewed interest in the luminosity distance calculations, accounting for the inhomogeneities in the Universe (e.g., [13, 15, 22]). However, the status of the second-order calculations of the luminosity distance is even worse than the linear-order case, as the calculations are much more involved and an explicit check of its gauge invariance is implausible. While there exist several groups in literature that have performed the second-order calculations, there is practically little hope that these calculations can be compared in a meaningful way to reach a consensus, as their notations and approaches are vastly different.

Here we provide a unified treatment of the luminosity distance in cosmology, comparing four different approaches in literature and critically assessing the sanity of the methods. This work will serve as a first step to go beyond the linear order in computing the luminosity distance and to build a coherent theoretical framework for all practitioners. The organization of the paper is as follows. We introduce our notation convention and present the distortion of a photon path from a straight line in Section 2. The distortion of the photon path is further decomposed into the radial and the angular deviations, and the observed redshift is related to the time coordinate of the source and its residual deviation. In Section 3, we use these geometric deviations to express the fluctuation in the luminosity distance. In Section 4, the optical scalar equation is derived to model the luminosity distance, and its relation in the conformally transformed metric is carefully derived. With such relations, the work by Sasaki [1] and Umeh et al. [2, 11] is derived with proper corrections and is then related to our geometric approach. In Section 5, the geodesic deviation equation is presented, and its counterpart in the conformally transformed metric is derived. We use the Jacobi mapping approach to derive the fluctuation in the luminosity distance and compare the result to the work in Bonvin et al. [3]. In Section 6, we present the basics of the geodesic light cone approach to the luminosity distance and provide corrections to its boundary condition. The luminosity distance with such corrections is consistent with that from other approaches. Finally, we conclude and discuss the implications in Section 7.

In this paper, we use the indices $a, b, c, d, \cdots$ to represent the four-dimensional spacetime indicies, while we use indicies $i, j, k, \cdots$ to represent the three-dimensional space indicies. In certain cases, we also use indices $I, J, K, \cdots$ in capital letters to represent the angular indicies in a spherical coordinate.

2 Light Propagation in Curved Spacetime

In this section, we present the metric convention and derive the propagation of light in a FRW universe. The real position of the source is decomposed into the apparent position inferred by the observed redshift and angular position and the residual perturbations in the position, providing basic ingredients for the geometric approach in Sec. 3.

2.1 Metric Convention and Photon Wavevector

Being an observable quantity, the luminosity distance is a gauge-invariant quantity, and it can be computed in any choice of gauge conditions. However, its computations in literature are predominantly performed in the conformal Newtonian gauge. To facilitate the comparison among various methods, we hereafter adopt the conformal Newtonian gauge for our metric representation:

$$ds^2 = g_{ab}dx^a dx^b = -a^2(1+2\psi)d\eta^2 + a^2(1+2\phi)\bar{g}_{ij}dx^i dx^j \ , \tag{2.1}$$

where the conformal time coordinate is η, the expansion scale factor is $a(\eta)$, and $\bar{g}_{ij}$ is the 3-spatial metric tensor in the background. Throughout the paper, we will consider two scalar degrees ψ and ϕ of freedom at the linear order in perturbations, ignoring the vector and tensor perturbations in the metric. As we are concerned with light propagation ($ds^2 = 0$), it is convenient to consider a conformal transformation:

$$ds^2 = g_{ab}dx^a dx^b = a^2 \hat{g}_{ab}dx^a dx^b \ , \tag{2.2}$$

where we used a hat to denote quantities in the conformally transformed metric $\hat{g}_{ab}$. The coordinates x^a of photon paths are identical in a given geometry, regardless of whether the physical metric g_{ab} or the conformally transformed metric $\hat{g}_{ab}$ is used, greatly simplifying the calculations when the latter is used. The four velocity u^a of timelike flows ($-1 = u^a u_a$) can be parametrized to the linear order in perturbations as

$$u^a = \frac{dx^a}{dt} = \frac{1}{a}\left(1-\psi \ , \ V^i\right) \ , \qquad u_a = -a\left(1+\psi \ , -V_i\right) \ , \tag{2.3}$$

where the proper time of the flows is dt and 3-spatial velocity V^i is based on 3-spatial metric $\bar{g}_{ij}$. With respect to the conformally transformed metric, the four velocity of the timelike flows is then

$$\hat{u}^a = a u^a = \left(1-\psi \ , \ V^i\right) \ , \qquad \hat{u}_a = \hat{g}_{ab}\hat{u}^b = \frac{u_a}{a} = -\left(1+\psi \ , -V_i\right) \ . \tag{2.4}$$

The light propagation is described by two key observable quantities measured by the observer in the rest frame: its phase ϑ and propagation direction $\boldsymbol{n}$. Given these two observables, the light cone in spacetime is defined as the two-dimensional surface with a constant phase ϑ, and the propagation direction $\boldsymbol{k}$ is orthogonal to the constant hypersurface of phase:

$$\vartheta = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t \ , \qquad k^a_L = \eta^{ab}\vartheta_{,b} = (\omega \ , \ \boldsymbol{k}) \ , \tag{2.5}$$

where the angular frequency is $\omega = 2\pi\nu$ in relation to its wavelength $\lambda = 1/\nu$ and we used the subscript L to emphasize that the four vector is written in the local rest frame with Minkowsky metric η_{ab}. For later convenience, we will use the observed direction of the light propagation $\boldsymbol{n} \equiv -\boldsymbol{k}/|\boldsymbol{k}|$ (opposite to the propagation direction), where the null condition imposes $\omega = |\boldsymbol{k}|$. The photon wavevector expressed in terms of local observable quantities can be transformed to that in a FRW coordinate:

$$k^a = \frac{dx^a}{dx^b_L}k^b_L = [e_b]^a k^b_L = \frac{\omega}{a}\left(1-\psi-V_\parallel \ , -n^i + V^i + \phi\, n^i\right) \ , \tag{2.6}$$

where we defined the line-of-sight velocity $V_\parallel = V^i n_i$ and the orthonormal local tetrads for the metric transformation are

$$[e_t]^a = u^a = \frac{1}{a}\left(1-\psi \ , \ V^i\right) \ , \qquad [e_j]^a = \frac{1}{a}\left[V_j \ , \ \delta^i_j(1-\phi)\right] \ , \tag{2.7}$$

defining the proper time-direction and three spacelike four vectors of the observer in the FRW frame. The local tetrads were constructed by using the orthonormality condition ($\eta_{ab} = g_{ab}[e_c]^a[e_d]^b$, $c, d = t, x, y, z$). The photon wavevector k^a in the FRW frame is different from that k^a_L measured in the local rest frame, since the observer is moving and the gravitational redshift affects the photon energy.

For the computational convenience, the photon wavevector is again conformally transformed as

$$\hat{k}^a = \frac{dx^a}{d\lambda} = \mathbb{C}a^2 k^a \ , \tag{2.8}$$

where the photon path is parametrized by the affine parameter λ and the overall constant factor $\mathbb{C}$ reflects the multiplicative freedom in the parametrization.[2] With the expression in Eq. (2.6), the conformally transformed wavevector is

$$\hat{k}^a = (\mathbb{C}\omega a)\left(1-\psi-V_\parallel \ , -n^i + V^i + \phi\, n^i\right) \equiv \left(1+\delta\nu \ , -n^i - \delta n^i\right) \ , \tag{2.9}$$

[2]The photon wavevector in Eq. (2.6) has no such degree of freedom, since it is completely specified in terms of physical quantities. Only when conformally transformed, the wavevector $\hat{k}^a$ has an additional freedom.

where we defined the perturbations $(\delta\nu,\ \delta n^i)$ in the photon wavevector. In a homogeneous universe, the photon energy is redshifted $E = \hbar\omega \propto 1/a$ as the Universe expands, and hence the constant factor $(\mathbb{C}\omega a)$ can be effectively removed by choosing $\mathbb{C} = 1/\overline{(\omega a)}$. The choice of the normalization in an inhomogeneous universe may be made only at one spacetime point, as the combination (ωa) is *not* constant any more [8], and this residual perturbation affects the parametrization $(\delta\nu,\ \delta n^i)$ of the photon wavevector at the perturbation level.[3] Since the degree of freedom in the conformally transformed wavevector has no physical significance, the final expression of observable quantities is independent of the normalization choice. However, this choice $\overline{(\mathbb{C}\omega a)} = 1$ is the most convenient and widely adopted in literature, while the choice differs at the perturbation level. We leave the normalization choice unspecified at the perturbation level and show that the final results are indeed independent of the normalization condition.

Given the observer four velocity and the photon wavevector, we can define the observed photon vector in a FRW frame as

$$N^a \equiv \frac{k^a}{k^b u_b} + u^a = \frac{1}{a}\left(V_\parallel\ ,\ n^i - \phi\, n^i\right)\ , \qquad N^a N_a = 1\ , \qquad N^a u_a = 0\ , \tag{2.10}$$

and this vector becomes in the rest frame of the observer:

$$N^a_L = \frac{k^a_L}{(k\cdot u)_L} + u^a_L = (0\ ,\ \boldsymbol{n})\ , \tag{2.11}$$

properly describing the observed direction of the photon path. However, the observed photon vector N^a is written in a FRW frame, and its spatial component is different from the observed angle $\boldsymbol{n}$ in the observer rest frame. Furthermore, we can parallelly transport N^a along the photon path to describe the "observed" photon direction at each spacetime point of the photon path. With respect to the conformally transformed metric, the observed photon vector is

$$\hat{N}^a = aN^a = \frac{\hat{k}^a}{\hat{k}^b \hat{u}_b} + \hat{u}^a = \left(V_\parallel\ ,\ n^i - \phi\, n^i\right)\ , \qquad \hat{N}_a = \left(-V_\parallel\ ,\ n_i + \phi\, n_i\right)\ , \tag{2.12}$$

independent of the normalization $\mathbb{C}$.

2.2 Deviations in the Photon Path

Given the expression of the photon wavevector, we want to integrate it over the affine parameter to obtain the coordinate expression for the light propagation and relate it to the observable quantities such as the observed redshift and the luminosity distance. Eventually, this would provide connections to the different approaches to modeling the luminosity distance. The light propagates in a homogeneous universe without any distortion in path, and the integration of the photon wavevector over the affine parameter yields the propagation path at a given affine parameter:

$$\bar{x}^a_\lambda = \left(\bar{\eta}_\lambda\ ,\ \bar{x}^i_\lambda\right) = \bar{x}^a_o + \int_0^\lambda d\lambda'\ \hat{\bar{k}}^a_{\lambda'} = \left(\bar{\eta}_o + \lambda,\ \ -\lambda\, n^i\right)\ , \tag{2.13}$$

where we set zero the affine parameter at the observer $\lambda_o = 0$. This background relation in turn defines the affine parameter in relation to the conformal time and the comoving distance as

$$\lambda = \bar{\eta}_\lambda - \bar{\eta}_o = -\bar{r}_\lambda\ , \qquad \bar{r}_z = \int_0^z \frac{dz'}{H(z')}\ , \qquad 1+z = \frac{1}{a(\bar{\eta}_z)}\ , \tag{2.14}$$

where the redshift parameter z is related to the affine parameter via the conformal time $\lambda_z = \bar{\eta}_z - \bar{\eta}_o$. We used a bar to represent quantities evaluated at λ in a homogeneous universe and those in an inhomogeneous universe will be represented without a bar.

[3] Depending on the choice of the normalization condition, the expressions of $\delta\nu$ and δn^i differ at each spacetime point, causing some confusion and difficulty in comparing results. However, only the background n^i will be used for the luminosity distance calculations at the linear order in perturbations.

The deviation from the straight path can be computed in a similar manner by integrating the perturbations $(\delta\nu, \delta n^i)$ in the photon wavevector over the affine parameter. These perturbations obey the geodesic equations, and the temporal and spatial components of the geodesic equations are

$$0 = \hat{k}^a \hat{k}^0{}_{;a} = \frac{d}{d\lambda}\delta\nu + \delta\Gamma^0 \ , \qquad 0 = \hat{k}^b \hat{k}^i{}_{;b} = \left(-n^{i\prime} + n^j n^i{}_{|j}\right) - \frac{d}{d\lambda}\delta n^i + \delta\Gamma^i \ , \tag{2.15}$$

where the background relation of the spatial component defines the photon path in a homogeneous universe as

$$0 = \frac{d}{d\lambda} = \bar{\hat{k}}^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial\eta} - n^j \frac{\partial}{\partial x^j} \ , \tag{2.16}$$

and we defined two metric perturbations in the geodesic equations [7, 8]

$$\delta\Gamma^0 \equiv \hat{\Gamma}^0_{ab}\hat{k}^a\hat{k}^b = \psi' - 2\psi_{,i}n^i + \phi' = 2\frac{d}{d\lambda}\psi - (\psi - \phi)' \ , \tag{2.17}$$

$$\delta\Gamma^i \equiv \delta\left(\hat{\Gamma}^i_{ab}\hat{k}^a\hat{k}^b\right) = \psi^{,i} - 2\phi' n^i + 2\phi_{,j}n^j n^i - \phi^{,i} = (\psi - \phi)^{,i} - 2n^i\frac{d}{d\lambda}\phi \ . \tag{2.18}$$

With the conformal transformation, the background relation of the geodesic equations is automatically satisfied. The perturbations to the photon wavevector can be formally integrated over the affine parameter as

$$\delta\nu_\lambda - \delta\nu_o = -\int_0^\lambda d\lambda' \ \delta\Gamma^0 = 2\psi_o - 2\psi_\lambda - \int_0^{\bar{r}_\lambda} d\bar{r} \ \ (\psi - \phi)' \ , \tag{2.19}$$

$$\delta n^i_\lambda - \delta n^i_o = \int_0^\lambda d\lambda' \ \delta\Gamma^i = -2n^i\left(\phi_\lambda - \phi_o\right) - \int_0^{\bar{r}_\lambda} d\bar{r} \ \ (\psi - \phi)^{,i} \ , \tag{2.20}$$

where we replaced the integration over the affine parameter λ with the integration over the comoving distance $\bar{r}$, representing the background photon path. However, it is noted that the integration over the affine parameter $(d\lambda)$ represents the evaluation of the integrands along the photon path x^a_λ, not along the background path $\bar{x}^a_\lambda$, although we will only need to consider the background path, as the integrands are already at the linear order in perturbations.

One further integration of the perturbations in the photon wavevector yields the deviation of the photon path from the background relation:

$$\delta x^a_\lambda - \delta x^a_o = \left(\delta\eta_\lambda - \delta\eta_o \ , \ \delta x^i_\lambda\right) = \left(\int_0^\lambda d\lambda' \ \delta\nu \ , \ \ -\int_0^\lambda d\lambda' \ \delta n^i\right) \ , \tag{2.21}$$

$$\delta\eta_\lambda - \delta\eta_o = -\bar{r}_\lambda\left(2\psi + \delta\nu\right)_o + \int_0^{\bar{r}_\lambda} d\bar{r}\Big[2\psi + (\bar{r}_\lambda - \bar{r})(\psi - \phi)'\Big] \ , \tag{2.22}$$

$$\delta x^i_\lambda = \bar{r}_\lambda\left(\delta n^i + 2\phi \ n^i\right)_o - \int_0^{\bar{r}_\lambda} d\bar{r}\left[2\phi \ n^i + (\bar{r}_\lambda - \bar{r})\left(\psi - \phi\right)^{,i}\right] \ , \tag{2.23}$$

where the spatial position at the observer can always be set zero $(x^i_o = 0)$ due to symmetry and the conformal time at the observer in an inhomogeneous universe deviates from its background value $\bar{\eta}_o$ by

$$\delta\eta_o = -\int_0^{\bar{\eta}_o} d\eta\,(a\psi) = \int_0^\infty dz \ \frac{\psi_o(z)}{H(z)} \ , \qquad \bar{\eta}_0 = \int_0^\infty \frac{dz}{H(z)} \ . \tag{2.24}$$

The light propagation is now expressed in terms of the metric perturbations and the perturbations $(\delta\nu, \delta n^i)$ to the photon wavevector at the observer position (or at some spacetime point). Since the photon wavevector follows the null geodesic, these perturbations are subject to the null condition:

$$0 = \hat{k}^a\hat{k}_a = 2\left(n^i\delta n_i - \delta\nu - \psi + \phi\right) \ , \tag{2.25}$$

and the remaining degrees of freedom in these perturbations $(\delta\nu, \delta n^i)$ are eliminated by the normalization condition in the conformal transformation in Eq. (2.9). Therefore, the photon path is completely specified given the metric perturbations and is independent of our parametrization of the photon wavevector.

Given the observed photon direction $\boldsymbol{n}$, it proves convenient to decompose the deviation of the photon path into one along the line-of-sight direction and one perpendicular to it:

$$\delta r_\lambda \equiv n_i\delta x^i_\lambda = -\int_0^\lambda d\lambda' n_i\delta n^i = \delta\eta_o - \delta\eta_\lambda + \int_0^{\bar r_\lambda} d\bar r\ (\psi-\phi)\ , \tag{2.26}$$

$$\delta x^\perp_\lambda = \bar r_\lambda(\delta\theta_\lambda,\ \sin\theta\ \delta\phi_\lambda) \equiv \delta x^i_\lambda n_{\perp i} = \bar r_\lambda \delta n^i_o n_{\perp i} - \int_0^{\bar r_\lambda} d\bar r\ \left(\frac{\bar r_\lambda - \bar r}{\bar r}\right)\hat\nabla(\psi-\phi)\ , \tag{2.27}$$

where the line-of-sight direction is $\boldsymbol{n}=(\theta,\phi)$ in spherical coordinates, $n^i_\perp$ is a direction perpendicular to $\boldsymbol{n}$, the angular gradient is $\hat\nabla$, and the perpendicular component of δx^i_λ is also characterized in spherical coordinates by using $(\delta\theta,\delta\phi)_\lambda$. These two components vanish at the observer position $\lambda=0$. Given the normalization condition in Eq. (2.9), the first component in $\delta x^\perp_\lambda$ is

$$\bar r_\lambda\delta n^i_o n_{\perp i} = -\bar r_\lambda V^i_o n_{\perp i}\ , \tag{2.28}$$

independent of the normalization condition at the perturbation order.

2.3 Observed Redshift and Geometric Distortions

An important observable in large-scale structure probes is the redshift of luminous sources such as galaxies. The photon wavelength is stretched due to the cosmic expansion, as it propagates throughout the Universe, and the observed redshift parameter is simply a measure of how much it has been stretched from its emission in the source rest frame along its journey to reach the observer:

$$1+z\equiv\frac{\lambda_o}{\lambda_s} = \frac{\omega_s}{\omega_o} = \frac{(k^a u_a)_s}{(k^a u_a)_o}\ , \tag{2.29}$$

where we used that the angular frequency ω measured in the rest frames of the source and the observer is a Lorentz scalar

$$\omega = -\eta_{ab}u^a_L k^b_L = -g_{ab}u^a k^b\ . \tag{2.30}$$

Using the conformally transformed wavevector, we can define a useful quantity $\widehat{\Delta\nu}$ that vanishes in the background in relation to the normalization condition in Eq. (2.9):

$$-\hat k^a\hat u_a = -\mathbb{C}a(k^a u_a) = \mathbb{C}a\omega \equiv 1+\widehat{\Delta\nu}\ , \qquad \widehat{\Delta\nu}\equiv\delta\nu+\psi+V_\parallel = n_i\delta n^i+\phi+V_\parallel\ , \tag{2.31}$$

and the observed redshift is then

$$1+z = \frac{a_o}{a_s}\frac{(\hat k^a\hat u_a)_s}{(\hat k^a\hat u_a)_o} = \frac{a_o}{a_s}\frac{1+\widehat{\Delta\nu}_s}{1+\widehat{\Delta\nu}_o}\ . \tag{2.32}$$

It is apparent that the observed redshift is affected not only by the cosmic expansion, but also by the line-of-sight velocity and the gravitational redshift, in the presence of inhomogeneities. To separate the background and the perturbation quantities, we define the distortion δz in the redshift as

$$1+z\equiv\frac{1+\delta z}{a_s}\ , \qquad \delta z = \mathcal{H}_o\delta\eta_o + \widehat{\Delta\nu}_s - \widehat{\Delta\nu}_o\ , \tag{2.33}$$

where we noted that $\eta_o=\bar\eta_o+\delta\eta_o$ and $a(\bar\eta_o)=1$. Using the relation for $\delta\nu$ in Eq. (2.19), the distortion in the redshift can be readily computed as [23]

$$\delta z = \mathcal{H}_o\delta\eta_o + \left(\psi - V_\parallel\right)_o - \left(\psi-V_\parallel\right)_z - \int_0^{\bar r_z} d\bar r\ (\psi-\phi)'\ , \tag{2.34}$$

where we replaced the source position denoted by the subscript s with the redshift parameter at the linear order in perturbations. It is noted that the expression is independent of our parametrization of the photon wavevector and the conformal transformation.

In the cosmological context, the redshift parameter is the only physically meaningful way to characterize its distance from the observer. Other parameters such as the affine parameter or the coordinate positions are both gauge-dependent and unobservable quantities, inadequate to describe physical quantities. To accommodate this point in our previous calculations, we define a series of perturbation quantities with respect to the background quantities evaluated at the observed redshift. In particular, they concern with the relation of the apparent and real positions of the source, representing the geometric distortions in an inhomogeneous universe: First, the affine parameter at the source position is split into λ_z corresponding to its observed redshift and the residual perturbation $\Delta\lambda_s$:

$$\lambda_s \equiv \lambda_z + \Delta\lambda_s \ , \qquad \lambda_z = \bar{\eta}_z - \bar{\eta}_o \ , \qquad 1 + z = \frac{1}{a(\bar{\eta}_z)} \ . \tag{2.35}$$

Second, the time coordinate η_s of the source is also split into one $\bar{\eta}_z$ associated with the redshift and the residual perturbation $\Delta\eta_z$:

$$\eta_s = \bar{\eta}_s + \delta\eta_s \equiv \bar{\eta}_z + \Delta\eta_z \ , \qquad \Delta\eta_z = \bar{\eta}_s - \bar{\eta}_z + \delta\eta_s = \Delta\lambda_s + \delta\eta_s \ . \tag{2.36}$$

Given the definition of the distortion δz of the redshift, the deviation $\Delta\eta_z$ in the time coordinate is further related to the distortion in the redshift as

$$\Delta\eta_z = \frac{\delta z}{\mathcal{H}} \ . \tag{2.37}$$

Third, we might define the deviation Δx^i_z in the spatial coordinates of the source, in a similar way $\Delta\eta_z$ is defined, but Δx^i_z will not appear in any of our equations at the linear order in perturbations. Finally, the geometric deviations $(\delta r_\lambda, \delta x^\perp_\lambda)$ of the photon path also need to be expressed with respect to the background position $\bar{x}^i_z = \bar{r}_z n^i$ at the observed redshift:

$$\delta r_z \equiv n_i x^i_s - \bar{r}_z = \delta r_s - \Delta\lambda_s = \delta\eta_o - \frac{\delta z}{\mathcal{H}} + \int_0^{\bar{r}_z} d\bar{r}\ (\psi - \phi) \ , \qquad \delta x^\perp_z = \delta x^\perp_s \ , \tag{2.38}$$

where the deviation perpendicular to the line-of-sight is unaffected as there is no transverse component in the background. At the observer position $\lambda = 0$, these perturbation quantities become

$$\text{at } \lambda = 0 \ (z = 0) \ : \quad \delta z = \mathcal{H}_0 \delta\eta_o \ , \quad \Delta\lambda_s = 0 \ , \quad \Delta\eta_z = \delta\eta_z = \delta\eta_o \ , \quad \delta r_z = \delta x^\perp_z = 0 \ . \tag{2.39}$$

2.4 Gravitational Lensing Magnification

Gravitational lensing describes the angular distortion in the photon path. The photon path is affected by the matter fluctuation and the metric perturbations along its path, and the observed angle of the source is non-trivially related to the position we would measure in the absence of such perturbations (see, e.g., [24, 25]). However, at the linear order in perturbations, the deviation from its unperturbed path is rather simple and well studied in literature. In view of our geometric description of the light propagation, gravitational lensing deals with the angular distortion $(\delta\theta, \delta\phi)$ without concerning the radial distortion δr in our formalism. In particular, only the lensing magnification will be needed in our application to the luminosity distance.

In general, the angular distortion in gravitational lensing is described by the inverse magnification matrix, or the deformation matrix $\mathbb{D}$, providing the relation between the observed angular position (θ, ϕ) at the observer to the angular position $(\theta + \delta\theta, \phi + \delta\phi)$ of the source. The deformation matrix is conventionally decomposed as

$$\mathbb{D} \equiv \frac{\partial(\theta + \delta\theta, \phi + \delta\phi)}{\partial(\theta, \phi)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \kappa + \gamma_1 & w + \gamma_2 \\ -w + \gamma_2 & \kappa - \gamma_1 \end{pmatrix} \ , \tag{2.40}$$

where the deviation from the identity captures the angular distortion in gravitation lensing with the convergence κ, the rotation w, and the shear (γ_1, γ_2). Since the surface brightness is conserved, the geometric enhancement of the solid angle $d\Omega_s$ of the source results in the gravitational lensing magnification, and it is the inverse Jacobian of the deformation matrix:

$$\frac{d\Omega_o}{d\Omega_s} = \frac{1}{\det \mathbb{D}} = \frac{1}{(1-\kappa)^2 - \gamma^2 + w^2} \simeq 1 + 2\,\kappa + \mathcal{O}(2) \;. \tag{2.41}$$

At the linear order in perturbation, the lensing magnification is proportional to the convergence. Given the definition of the deformation matrix, its determinant can be computed as [8]

$$\det \mathbb{D} = \frac{\sin(\theta+\delta\theta)}{\sin\theta}\left[1 + \frac{\partial}{\partial\theta}\delta\theta + \frac{\partial}{\partial\phi}\delta\phi + \frac{\partial}{\partial\theta}\delta\theta\frac{\partial}{\partial\phi}\delta\phi - \frac{\partial}{\partial\theta}\delta\phi\frac{\partial}{\partial\phi}\delta\theta\right] \;, \tag{2.42}$$

valid to *all orders* in perturbations, when the angular distortion $(\delta\theta, \delta\phi)$ is also defined non-perturbatively. The lensing convergence at the linear order is, therefore, derived as

$$\kappa = -\frac{1}{2}\left[\left(\cot\theta + \frac{\partial}{\partial\theta}\right)\delta\theta + \frac{\partial}{\partial\phi}\delta\phi\right] = -V_{\parallel o} + \int_0^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}_z\bar{r}}\right)\hat{\nabla}^2\left(\frac{\psi-\phi}{2}\right) \;, \tag{2.43}$$

where the line-of-sight velocity at the observer position arises from Eq. (2.28) and the angular Laplacian is

$$\hat{\nabla}^2 = \left(\cot\theta + \frac{\partial}{\partial\theta}\right)\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \;. \tag{2.44}$$

The line-of-sight velocity at the observer position is often ignored in the gravitational lensing convergence. Consequently, it is missing in the luminosity distance calculation as well [13, 26, 27]. In [15], all the observer velocity contributions (in δz, δr, and κ) are ignored by hand (see [28] for interesting discussion of the velocity contribution to the luminosity distance).

3 Geometric Approach to the Luminosity Distance

The apparent source position $\bar{x}^a_s$ is inferred by using the observable quantities such as the observed redshift and the observed angular position. In Sec. 2, the distortion in the apparent source position compared to the real source position is geometrically decomposed with the radial distortion δr and the angular distortion κ, accounting for the distortion δz in the observed redshift as the "distortion" in the time direction. Since gravity achromatically affects the light propagation, our geometric approach should provide a good physical description of cosmological probes including the luminosity distance fluctuations.

The crucial element in the geometric approach is the relation of the source position to the physical quantities. In [18, 19], the covariant expression was first applied in cosmology to compute the physical volume occupied by the source x^a_s with its inferred position $\bar{x}^a_s$. Such covariant expression is essential to ensuring the gauge-invariance of the expression of a physical volume in terms of the observable quantities [19]. Adopting this approach, Jeong et al. [6] first presented a covariant expression of the physical area, occupied by the source x^a_s with its inferred position $\bar{x}^a_s$, and the area is further defined as one perpendicular to the photon propagation. This altogether provides a key element for computing the luminosity distance in the geometric approach.

3.1 Covariant Expression

The luminosity distance is related to the angular diameter distance by the reciprocity relation

$$\mathcal{D}_A(z) = \frac{\mathcal{D}_L(z)}{(1+z)^2} \;, \tag{3.1}$$

at the given observed redshift, and this relation is exact to all orders in perturbations. Therefore, the fluctuation in the angular diameter distance is equivalent to that in the luminosity distance:

$$1 + \delta\mathcal{D}_L = \frac{\mathcal{D}_L(z)}{\bar{D}_L(z)} = \frac{\mathcal{D}_A(z)}{\bar{D}_A(z)} = 1 + \delta\mathcal{D}_A \;, \tag{3.2}$$

providing a simple way to derive the luminosity distance using our description of the geometric distortion. The angular diameter distance is the distance, at which a unit area $d\mathcal{A}$ at the observed redshift z is subtended by a solid angle $d\Omega_o$ at the observer:

$$d\mathcal{A} = \mathcal{D}_A^2(z) d\Omega_o \ . \tag{3.3}$$

Since the area $d\mathcal{A}$ in the source rest frame is perpendicular to the four velocity u^a at the source and also to the observed photon vector N^a transported to the source position, the area spanned by the observed angle can be computed in a covariant way [6–8] as

$$d\mathcal{A} = \sqrt{-g}\ \varepsilon_{dabc} u^d N^a \frac{\partial x^b}{\partial\theta} \frac{\partial x^c}{\partial\phi}\ d\theta\ d\phi \ , \tag{3.4}$$

where ε_{abcd} is the Levi-Civita symbol with $\varepsilon_{0123} = 1$. Using the relations

$$a_s = \frac{1+\delta z}{1+z} \ , \qquad \sqrt{-g} = a_s^4 (1+\delta g) \ , \qquad \delta g = \psi + 3\phi \ , \qquad \bar{D}_A(z) = \frac{\bar{r}_z}{1+z} \ , \tag{3.5}$$

we can re-arrange the the covariant equation by

$$\mathcal{D}_A^2(z) = \bar{D}_A^2(z) \frac{(1+\delta g)(1+\delta z)^2}{\bar{r}_z^2 \sin\theta} \left(\varepsilon_{dabc} \hat{u}^d \hat{N}^a \frac{\partial x^b}{\partial\theta} \frac{\partial x^c}{\partial\phi} \right) \ , \tag{3.6}$$

and the round bracket can be further expanded in its Levi-Civita indices as

$$\left(\varepsilon_{dabc} \hat{u}^d \hat{N}^a \frac{\partial x^b}{\partial\theta} \frac{\partial x^c}{\partial\phi} \right) = \varepsilon_{0ijk} \hat{N}^i \frac{\partial x^j}{\partial\theta} \frac{\partial x^k}{\partial\phi} + \varepsilon_{0ijk} (-\psi) \hat{N}^i \frac{\partial x^j}{\partial\theta} \frac{\partial x^k}{\partial\phi} + \varepsilon_{kabc} V^k \hat{N}^a \frac{\partial x^b}{\partial\theta} \frac{\partial x^c}{\partial\phi} \ , \tag{3.7}$$

where the third component is already at the second order in perturbations. Since the second component is already at the linear order, we need to evaluate the remaining term at the background level:

$$\varepsilon_{0ijk} (-\psi) \hat{\bar{N}}^i \frac{\partial \bar{x}^j}{\partial\theta} \frac{\partial \bar{x}^k}{\partial\phi} = -\psi\ \boldsymbol{n} \cdot \left(\bar{r}_z \frac{\partial}{\partial\theta} \boldsymbol{n} \times \bar{r}_z \frac{\partial}{\partial\phi} \boldsymbol{n} \right) = -\psi\ \bar{r}_z^2 \sin\theta \ . \tag{3.8}$$

Similarly, the first component can be computed by decomposing it according to the perturbation orders as

$$\begin{aligned} \varepsilon_{0ijk} \hat{N}^i \frac{\partial x^j}{\partial\theta} \frac{\partial x^k}{\partial\phi} &= \bar{r}_z^2 \sin\theta + \varepsilon_{0ijk} \hat{N}^{i(1)} \frac{\partial \bar{x}^j}{\partial\theta} \frac{\partial \bar{x}^k}{\partial\phi} + \varepsilon_{0ijk} \hat{\bar{N}}^i \left(\frac{\partial \delta x^j}{\partial\theta} \frac{\partial \bar{x}^k}{\partial\phi} + \frac{\partial \bar{x}^j}{\partial\theta} \frac{\partial \delta x^k}{\partial\phi} \right) \\ &= \bar{r}_z^2 \sin\theta\ \ (1-\phi) + \bar{r}_z \boldsymbol{n} \cdot \left(\frac{\partial}{\partial\theta} \delta\boldsymbol{x} \times \frac{\partial}{\partial\phi} \boldsymbol{n} + \frac{\partial}{\partial\theta} \boldsymbol{n} \times \frac{\partial}{\partial\phi} \delta\boldsymbol{x} \right) = \bar{r}_z^2 \sin\theta\ \left(1 - \phi + 2\ \frac{\delta r_z}{\bar{r}_z} - 2\ \kappa \right) \ , \end{aligned} \tag{3.9}$$

where the vector product is essentially the distortion in the geometric volume. Therefore, the fluctuation in the angular diameter distance amounts to

$$\delta\mathcal{D}_A = \frac{1}{2} \delta g + \delta z + \frac{1}{2} \left(-\psi - \phi + 2\ \frac{\delta r_z}{\bar{r}_z} - 2\ \kappa \right) = \phi + \delta z + \frac{\delta r_z}{\bar{r}_z} - \kappa \ , \tag{3.10}$$

and it is equivalent to the fluctuation in the luminosity distance $\delta\mathcal{D}_A = \delta\mathcal{D}_L$. The fluctuations arise because of the (comoving) volume distortions decomposed into the radial component δr and the angular component κ, the relation to the proper volume by the metric determinant δg, and finally the use of the observed redshift δz in the luminosity distance. We will use our geometric approach to connect different methods for computing the luminosity distance in the following sections.

3.2 Standard Ruler

Another approach to computing the physical area was developed in [29] under the name of "cosmic ruler." The idea is to relate the (known) scale of a standard ruler placed at source x^a_s to the inferred position $\bar{x}^a_s$ in terms of the geometric distortions. Similar in spirit to obtaining the covariant expression in Eq. (3.4), this method computes the length of a standard ruler with two end points of the ruler described by the observed angle (θ, ϕ) and $(\theta + d\theta, \phi + d\phi)$. The original method [29] computes the distortion in three-dimensional space, but here we will focus on one that is relevant to computing the luminosity distance.

Assuming two ends points x^a_1 and x^a_2 of a standard ruler are so close to each other that they both are at the same observed redshift z, the difference $\delta\bar{x}^a$ in the inferred positions can be computed as

$$\bar{x}^a_1 = [\bar{\eta}_z,\ \bar{r}_z n^i] \ , \qquad \bar{x}^a_2 = [\bar{\eta}_z,\ \bar{r}_z(n^i + \Delta n^i)] \ , \qquad \delta\bar{x}^a \equiv \bar{x}^a_1 - \bar{x}^a_2 = (0, -\bar{r}_z\Delta n^i) \ , \tag{3.11}$$

where the difference in the observed angle $\Delta n^i = (d\theta, d\phi)$ in a spherical coordinate will be taken to be zero. Since two apparent angular directions are unit vectors, we have $\delta\bar{x}^a \propto \Delta n^i \perp n^i$. Given the apparent positions of the standard ruler, we can derive the relation of its apparent scale R_z to the known scale of the ruler R:

$$\begin{aligned} R^2 &= \mathcal{P}_{ab}(x_1 - x_2)^a(x_1 - x_2)^b = \mathcal{P}_{ab}\left(\delta\bar{x}^a\delta\bar{x}^b + 2\delta\bar{x}^a\Delta x^b + \Delta x^a\Delta x^b\right) \\ &= R^2_z(1 + 2\ \delta z) + 2a^2(\bar{\eta}_z)\left(\phi\ \bar{g}_{ij}\delta\bar{x}^i\delta\bar{x}^j + \bar{g}_{ij}\delta\bar{x}^i\Delta x^j\right) \ , \end{aligned} \tag{3.12}$$

where we ignored the second order terms, the apparent scale of the ruler is related to the angular diameter distance in a homogeneous universe

$$R^2_z \equiv a^2(\bar{\eta}_z)\bar{g}_{ij}\delta\bar{x}^i\delta\bar{x}^j = \bar{D}^2_A(z)\bar{g}_{ij}\Delta n^i\Delta n^j \ , \tag{3.13}$$

the difference Δx^a in the real and the apparent positions of two end points are

$$x^a_1 = \bar{x}^a_1 + \Delta x^a_1 \ , \qquad x^a_2 = \bar{x}^a_2 + \Delta x^a_2 \ , \qquad \Delta x^a \equiv \Delta x^a_1 - \Delta x^a_2 \ , \tag{3.14}$$

and the projection tensor is

$$\mathcal{P}_{ab} = g_{ab} + u_a u_b \ , \qquad \mathcal{P}^a_a = 3 \ . \tag{3.15}$$

Two end points are projected to a hypersurface orthogonal to u^a, because the standard ruler is defined in a local rest frame of an observer u^a. With the approximation that the scale of the ruler is small, the projection tensor $\mathcal{P}_{ab}$ at x^a_1 was used without ambiguity. Similarly, no ambiguity arises in the line-of-sight direction $(= \boldsymbol{n}_1)$.

It is apparent in Eq. (3.12) that the scale of the ruler is set by its apparent angular scale $R^2 \propto \Delta n^2$, except the last term with Δx^i. Since Δx^i vanishes as the angular scale $\Delta n^i = (d\theta, d\phi)$ is taken to be zero, the leading contribution in proportion to Δn^i can be computed as

$$\Delta x^i \simeq \delta\bar{x}^j \frac{\partial}{\partial x^j}\Delta x^i \simeq -\bar{r}_z\Delta n^j \frac{\partial}{\partial x^j}\Delta x^i \simeq -\bar{r}_z\Delta n^j \frac{\partial}{\partial x^j}\left(n^i\delta r + \delta x^\perp_z\right) + \mathcal{O}(\Delta n^2) \ , \tag{3.16}$$

and the last term in Eq. (3.12) then becomes

$$2a^2(\bar{\eta}_z)\bar{g}_{ij}\delta\bar{x}^i\Delta x^j = \bar{D}^2_A(z)\left[2\ \frac{\delta r}{\bar{r}_z}\bar{g}_{ij}\Delta n^i\Delta n^j + \bar{g}_{ij}\Delta n^i\Delta n^k\hat{\nabla}_k(\delta\theta,\ \sin\theta\ \delta\phi)^j\right] \ , \tag{3.17}$$

where we took the limit $x^a_2 \to x^a_1$ and the component is expressed in spherical coordinates. Therefore, putting it altogether, the relation in Eq. (3.12) can be used to compute a unit physical area $d\mathcal{A}$ spanned by the angular scale $(d\theta, d\phi)$ as

$$d\mathcal{A} = \bar{D}^2_A(z)\left(1 + 2\ \delta z + 2\ \phi + 2\ \frac{\delta r_z}{\bar{r}_z} - 2\ \kappa\right) d\Omega = \mathcal{D}^2_A(z)\ d\Omega \ , \tag{3.18}$$

consistent with the result in previous section.

The covariant expression in Sec. 3.1 directly computes a unit physical area $d\mathcal{A}$ in an observer rest frame u^a, corresponding to the angular scale $(d\theta, d\phi)$. The standard ruler calculation in [29] starts with a scale in an observer rest frame, demanding that it be observed at the angular scale $(d\theta, d\phi)$. The resulting expressions are naturally described by the geometric distortions in both cases.[4]

4 Sachs Approach to the Luminosity Distance: Optical Scalar Equation

In this section, we present the optical scalar equation and its relation to the luminosity distance. This approach was first taken by Sasaki [1] and later extended by Umeh et al. [2, 11].

4.1 Optical Scalar Equation

The light propagation in an inhomogeneous universe was presented in Sec. 2. Now we consider a bundle of light rays and how its properties like the expansion θ and the shear σ propagate along the geodesic (see [31, 32] for review). In the rest frame of an observer described by the four velocity u^a, a unit area $d\mathcal{A}$ can be uniquely defined as one perpendicular to the photon propagation N^a, and the projection into this two-dimensional hypersurface is described by the projection tensor:

$$\mathcal{H}_{ab} \equiv g_{ab} + u_a u_b - N_a N_b = g_{ab} + \frac{k_a k_b}{(k\cdot u)^2} - 2\frac{u_{(a}k_{b)}}{k\cdot u} \ , \tag{4.1}$$

where the two-dimensional projection tensor is orthogonal to the observer and the light propagation direction

$$0 = \mathcal{H}_{ab}u^b = \mathcal{H}_{ab}N^b = \mathcal{H}_{ab}k^b \ , \qquad\qquad \mathcal{H}^a_a = 2 \ . \tag{4.2}$$

As the distortion tensor $k_{a;b}$ is three-dimensional, we project the distortion tensor into the two-dimensional hypersurface and decompose it in terms of the expansion θ and the projected shear σ_{ab} as

$$\mathcal{H}_{ac}\mathcal{H}_{bd}k^{c;d} = \frac{1}{2}\theta\ \mathcal{H}_{ab} + \sigma_{ab} \ , \tag{4.3}$$

where the expansion and the amplitude of the shear are independent of its projection tensor

$$\theta = \mathcal{H}^a_c\mathcal{H}^d_a k^c{}_{;d} = k^a{}_{;a} \ , \qquad\qquad 2\sigma^2 \equiv \sigma_{ab}\sigma^{ab} = k_{a;b}k^{a;b} - \frac{1}{2}\theta^2 \ . \tag{4.4}$$

When the photon wavevector is parametrized to be orthogonal to the hypersurface of constant phase ϑ, as in Eq. (2.6), there is no asymmetric part of the distortion tensor (or rotation). The shear vanishes in the background.

Parametrizing the photon wavevector $k^a = dx^a/d\Lambda$ with (physical) affine parameter Λ, the propagation of the distortion tensor can be readily computed as

$$\frac{D}{d\Lambda}\left(k^a{}_{;b}\right) = k^c k^a{}_{;bc} = -k_{c;b}k^{a;c} - R^a_{dbc}k^c k^d \ , \tag{4.5}$$

where we used the geodesic equation and the definition of the Riemann tensor. Contracting the two indices, we can derive the propagation equation for the expansion (or *Sachs equation* [33, 34])

$$\frac{d}{d\Lambda}\theta + \frac{1}{2}\theta^2 + 2\sigma^2 + R_{ab}k^a k^b = 0 \ , \tag{4.6}$$

[4]Schmidt and Jeong [29] further considers a case in which the scale of a standard ruler evolves in time, i.e., by replacing δz in Eq. (3.18) with

$$\delta z\left(1 - \frac{\partial \ln R^2}{\partial \ln a}\right) \ . \tag{3.19}$$

However, such term would break the gauge-invariance of the luminosity distance calculation, unless the frame of such evolution is specified, e.g., the proper-time hypersurface of an observer:

$$\delta z - \delta z_p \left(\frac{\partial \ln R^2}{\partial \ln a}\right) \ , \tag{3.20}$$

where the subscript indicates that a gauge choice is made, while δz without it is left unspecified as other terms in Eq. (3.18). This gauge issue is also resolved in their later work [30].

where it is noted that the propagation equation and the expansion itself are independent of a projection tensor $\mathcal{H}_{ab}$.

In particular, the expansion of light rays represents the change of a unit area $d\mathcal{A}$, swept by the bundle of neighboring light rays, and hence it is related to the angular diameter distance as in Eq. (3.3):

$$\theta = \frac{d}{d\Lambda} \ln d\mathcal{A} = \frac{d}{d\Lambda} \ln \mathcal{D}_A^2 \ . \tag{4.7}$$

A formal integration of the expansion over the affine parameter yields the angular diameter distance

$$\mathcal{D}_A(\Lambda) = \mathcal{D}_A(\epsilon) \exp\left[\frac{1}{2}\int_\epsilon^\Lambda d\Lambda' \ \theta(x^a_{\Lambda'})\right] \ , \tag{4.8}$$

where the affine parameter $\Lambda = \epsilon$ will be eventually taken to be zero, representing the observer position. The luminosity distance can be readily obtained by using the reciprocity relation in Eq. (3.1). As we have noted in Sec. 2, the calculations are greatly simplified, when we work in the conformally transformed metric, as it is devoid of the scale factor and its derivatives like the Hubble parameter. We will do so in deriving the above solution for the angular diameter distance.

4.2 Conformal Transformation and Angular Diameter Distance

Given the conformal transformation of the metric in Eq. (2.2), we have proved in Sec. 2 that some quantities between these two metrics are trivially related by the scale factor a, while some quantities are *not*:

$$g_{ab} = a^2 \hat{g}_{ab} \ , \qquad u^a = a\hat{u}^a \ , \qquad N^a = a\hat{N}^a \ , \qquad \hat{k}^a = \mathbb{C}a^2 k^a \ . \tag{4.9}$$

With the above relations, the projection tensor transforms trivially as $\mathcal{H}_{ab} = a^2\hat{\mathcal{H}}_{ab}$. We need to exercise caution in deriving the governing equations and the relevant quantities in the conformally transformed metric. For example, the expansion θ is transformed differently, as the covariant derivatives in two metrics are non-trivially related.

Noting that the metric tensors are parallelly transported in each metric

$$0 = \nabla_a g_{bc} = \hat{\nabla}_a \hat{g}_{bc} \ , \tag{4.10}$$

we can show that the covariant derivative $\hat{\nabla}_a$ in the conformally transformed metric, when acted on a four vector $\hat{k}^a$, is related to the covariant derivative ∇_a in the original metric as

$$\hat{\nabla}_a \hat{k}^b = \nabla_a \hat{k}^b + C^b_{ac}\hat{k}^c \ , \qquad\qquad C^c_{ab} = \mathcal{H}\left(g_{ab}g^{c0} - \delta^0_a\delta^c_b - \delta^c_a\delta^0_b\right) \ , \tag{4.11}$$

where C^b_{ac} is the symmetric tensor, connecting two derivative operators. Contracting two indices, the expansion $\hat{\theta}$ in the conformally transformed metric can be related to the expansion θ as

$$\hat{\theta} = \hat{\nabla}_a \hat{k}^a = \nabla_a\left(\mathbb{C}a^2 k^a\right) - 4\mathcal{H}\hat{k}^0 = \mathbb{C}a^2\theta - 2\mathcal{H}\hat{k}^0 = \frac{d\Lambda}{d\lambda}\,\theta - \frac{d}{d\lambda}\ln a^2 \ . \tag{4.12}$$

Using the relation of the expansion to a unit area in Eq. (4.7), we derive its counterpart in the conformally transformed metric

$$\hat{\theta} = \frac{d}{d\lambda}\ln\left(\frac{d\mathcal{A}}{a^2}\right) = \frac{d}{d\lambda}\ln\left(\frac{\mathcal{D}_A}{a}\right)^2 \ , \tag{4.13}$$

and therefore, the angular diameter distance can be obtained by using $\hat{\theta}$ as

$$\mathcal{D}_A(\lambda) = \frac{a(\lambda)}{a(\epsilon)}\,\mathcal{D}_A(\epsilon)\exp\left[\frac{1}{2}\int_\epsilon^\lambda d\lambda' \ \hat{\theta}(x^a_{\lambda'})\right] \ . \tag{4.14}$$

The propagation of the expansion $\hat{\theta}$ should be governed by one similar to the Sachs equation. Indeed, the derivation of the Sachs equation is generic, such that it only depends on the parametrization of the photon wavevector, the metric tensor and its derivatives. Therefore, it is obvious that the Sachs equation takes the same form in the conformally transformed metric:

$$\frac{d}{d\lambda}\hat{\theta} + \frac{1}{2}\hat{\theta}^2 + 2\hat{\sigma}^2 + \hat{\mathfrak{R}} = 0 \ , \qquad \hat{\mathfrak{R}} \equiv \hat{R}_{ab}\hat{k}^a\hat{k}^b \ , \tag{4.15}$$

where the expansion and the shear are defined in the same way, but in terms of the conformally transformed wavevector $\hat{k}^a$.

4.3 Sasaki 1987 [1]

In the pioneering paper [1], the complete derivation of the luminosity distance was presented with the general metric representation, and the gauge invariance of its expression was shown by explicitly transforming each component under a general coordinate transformation. Despite its perfection, the work [1] remains as one of the most incomprehensible in the luminosity distance literature.

Here we reproduce the derivation of the luminosity distance in [1], paying particular attention to its connection to our geometric approach. Given the solution in Eq. (4.14), we need to compute the integration of the expansion and the angular diameter distance $\mathcal{D}_A(\epsilon)$ near the observer position. First, consider a bundle of light rays from a distant source converging at the observer position, where the expansion of light rays becomes infinite. The expansion at the background can be readily obtained by integrating the Sachs equation with the boundary condition as

$$\frac{d\hat{\bar{\theta}}}{d\lambda} = -\frac{1}{2}\hat{\bar{\theta}}^2 \ , \qquad \hat{\bar{\theta}} = \frac{2}{\lambda} \ , \tag{4.16}$$

where both $\hat{\sigma}$ and $\hat{R}_{ab}\hat{k}^a\hat{k}^b$ in the Sachs equation vanish at the background. The integration in the angular diameter distance in Eq. (4.14) trivially becomes

$$\exp\left[\frac{1}{2}\int_\epsilon^\lambda d\lambda' \ \hat{\bar{\theta}}\right] = \frac{\lambda}{\epsilon} \ , \tag{4.17}$$

where $\lambda = \epsilon$ will be taken to be zero.

Next, consider a local Lorentz frame of the observer to compute the relation of the angular diameter distance $\mathcal{D}_A(\epsilon)$ near the observer to the photon path. In the observer's rest frame, the angular diameter distance at $\lambda = \epsilon$ is essentially the physical distance Δr that light travels for an infinitesimal amount of time Δt, corresponding to the affine parameter $\lambda = \epsilon$:

$$\omega = -\eta_{ab}k^a_L u^b_L = k^0_L = \frac{dt}{d\lambda}\frac{1}{\mathbb{C}a^2} \ , \qquad \Delta t = (\mathbb{C}a^2\omega)_o\epsilon \ , \qquad \mathcal{D}_A(\epsilon) = |\Delta t| = -(\mathbb{C}a^2\omega)_o\epsilon \ . \tag{4.18}$$

Using this relation and expanding the integration in Eq. (4.14) to the linear order in perturbation, the angular diameter distance at the source λ_s can be obtained as

$$\begin{aligned} \mathcal{D}_A &= -a_s(\mathbb{C}a\omega)_o\lambda_s\left(1 + \frac{1}{2}\int_0^{\lambda_s} d\lambda \ \delta\hat{\theta}\right) = -\left(\frac{1+\delta z}{1+z}\right)\left(1 + \widehat{\Delta\nu}_o\right)(\lambda_z + \Delta\lambda_s)\left(1 + \frac{1}{2}\mathfrak{I}\right) \\ &= \frac{\bar{r}_z}{1+z}\left(1 + \delta z + \widehat{\Delta\nu}_0 + \frac{\Delta\lambda_s}{\lambda_z} + \frac{1}{2}\mathfrak{I}\right) = \bar{D}_A(1 + \delta\mathcal{D}_A) \ , \end{aligned} \tag{4.19}$$

where the infinitesimal affine parameter ϵ cancels, we replaced each term a_s, $(\mathbb{C}a\omega)_o$, and λ_s, and we defined the integration of the expansion perturbation $\delta\hat{\theta}$

$$\mathfrak{I} \equiv \int_0^{\lambda_s} d\lambda \ \delta\hat{\theta}(x^a_\lambda) \ . \tag{4.20}$$

The expansion perturbation $\delta\hat{\theta}$ is subject to the linear-order Sachs equation and its solution given the background solution $\hat{\bar{\theta}}$ is

$$\frac{d}{d\lambda}\delta\hat{\theta} + \hat{\bar{\theta}}\,\delta\hat{\theta} + \delta\hat{\mathfrak{R}} = 0 \,, \qquad \delta\hat{\theta}_\lambda = -\frac{1}{\lambda^2}\int_0^\lambda d\lambda'(\lambda')^2\delta\hat{\mathfrak{R}}_{\lambda'} \,. \tag{4.21}$$

As the Ricci scalar $\hat{R} = \hat{R}^a_a$ in the transformed metric vanishes in the background, the integrand for the expansion perturbation is

$$\begin{aligned}\delta\hat{\mathfrak{R}}_\lambda &= \hat{\mathfrak{R}}(x^a_\lambda) = \Delta\left(\psi-\phi\right) - \left(\psi-\phi\right)_{,i|j} n^i n^j - 2\frac{d^2}{d\lambda^2}\phi \\ &= \frac{2}{\bar{r}}(\psi-\phi)' - \frac{2}{\bar{r}}\frac{d}{d\lambda}(\psi-\phi) + \frac{1}{\bar{r}^2}\hat{\nabla}^2(\psi-\phi) - 2\frac{d^2}{d\lambda^2}\phi \,. \end{aligned} \tag{4.22}$$

Using the formal solution of $\delta\hat{\theta}_\lambda$, the integration of the expansion perturbation can be computed as

$$\mathfrak{I} = \int_0^{\lambda_s} d\lambda\;\delta\hat{\theta}_\lambda = -\int_0^{\lambda_s} d\lambda\left(\frac{\lambda_s-\lambda}{\lambda_s\lambda}\right)\lambda^2\delta\hat{\mathfrak{R}}_\lambda \equiv \mathfrak{I}_A + \mathfrak{I}_B + \mathfrak{I}_C + \mathfrak{I}_D \,, \tag{4.23}$$

where we split the integration into four components, corresponding to each component in the integrand in Eq. (4.22)

$$\begin{aligned}\mathfrak{I}_A &= -\frac{2}{\bar{r}_z}\int_0^{\bar{r}_z} d\bar{r}\;(\bar{r}_z-\bar{r})\,(\psi-\phi)' \,, \qquad \mathfrak{I}_B = 2(\psi-\phi)_o - \frac{2}{\bar{r}_z}\int_0^{\bar{r}_z} d\bar{r}\;(\psi-\phi) \,, \\ \mathfrak{I}_C &= -\int_0^{\bar{r}_z} d\bar{r}\left(\frac{\bar{r}_z-\bar{r}}{\bar{r}_z\bar{r}}\right)\hat{\nabla}^2(\psi-\phi) = -2\kappa - 2V_{\|_o} \,, \qquad \mathfrak{I}_D = 2(\phi+\phi_o) - \frac{4}{\bar{r}_z}\int_0^{\bar{r}_z} d\bar{r}\;\phi \,. \end{aligned} \tag{4.24}$$

So, the integration $\mathfrak{I}$ concerns the angular distortion κ in addition to several contributions of the gravitational potential. Using the relation for $\delta\eta_z - \delta\eta_o$ in Eq. (2.22), the residual perturbation $\Delta\lambda_s$ of the source λ_s is related to the radial distortion δr_z as

$$-\Delta\lambda_s = \delta\eta_z - \frac{\delta z}{\mathcal{H}} = \delta r_z - \bar{r}_z(2\psi+\delta\nu)_o + \int_0^{\bar{r}_z} d\bar{r}\left[(\psi+\phi) + (\bar{r}_z-\bar{r})(\psi-\phi)'\right] \,, \tag{4.25}$$

$$\frac{\Delta\lambda_s}{\lambda_z} = \frac{\delta r_z}{\bar{r}_z} - \widehat{\Delta\nu}_o + V_{\|_o} + \phi - \frac{1}{2}\left(\mathfrak{I}_A + \mathfrak{I}_B + \mathfrak{I}_D\right) \,. \tag{4.26}$$

Finally, putting it all together, we derive

$$\delta\mathcal{D} = \delta z + \widehat{\Delta\nu}_o + \frac{\Delta\lambda_s}{\lambda_z} + \frac{1}{2}\mathfrak{I} = \delta z + \frac{\delta r_z}{\bar{r}_z} - \kappa + \phi \,, \tag{4.27}$$

and the result is consistent with the previous calculations in our geometric approach.

In [1], the normalization convention was taken as $\widehat{\Delta\nu}_s = 0$ at the source position, but as we demonstrated, the final expression is independent of the normalization convention. After the complete derivation of the expression with the general metric, additional computation was performed [1] in the conformal Newtonian gauge, in which the coordinate lapse $\delta\eta_o$ at the observer position was removed by taking the angle average of $\delta\mathcal{D}_L$ and demanding it should vanish at $z = 0$:

$$0 \equiv \lim_{z\to 0}\int\frac{d\Omega}{4\pi}\;\delta\mathcal{D}_L(z) = \frac{3}{\eta_0}\delta\eta_0 + \psi_o + \frac{\eta_o^2}{4}\left(\psi_{,ij}n^i n^j\right)_o \,, \qquad \delta\eta_o \equiv -\frac{\eta_o}{3}\left[\psi + \frac{\eta^2}{18}\Delta\psi\right]_o \,. \tag{4.28}$$

This is in conflict with Eq. (2.24). The coordinate lapse at the observer position is important in ensuring the gauge-invariance of the full expression $\delta\mathcal{D}_L$, and it cannot be removed this way.

4.4 Umeh, Clarkson, Maartens 2014 [2]

Despite the difference in the master equations, the approach taken in [2, 11] is indeed identical to one in [1], all of which are based on the Sachs equation (4.15). Therefore, this method can reproduce the correct luminosity distance, while a few terms are neglected in [2, 11].

Given the relation of the expansion $\hat{\theta}$ to the angular diameter distance in Eq. (4.13), we take the derivative with respect to the affine parameter:

$$\frac{d}{d\lambda}\hat{\theta} = \frac{d^2}{d\lambda^2}\ln\left(\frac{\mathcal{D}_A}{a}\right)^2 = 2\,\frac{a}{\mathcal{D}_A}\frac{d^2}{d\lambda^2}\left(\frac{\mathcal{D}_A}{a}\right) - \frac{1}{2}\hat{\theta}^2 \ , \tag{4.29}$$

and using the Sachs equation to remove the expansion $\hat{\theta}$, we derive the differential equation for the angular diameter distance:

$$\frac{d^2}{d\lambda^2}\left(\frac{\mathcal{D}_A}{a}\right) = -\frac{1}{2}\,\hat{\mathfrak{R}}\left(\frac{\mathcal{D}_A}{a}\right) \ , \tag{4.30}$$

where we kept the terms up to the linear order in perturbations. Since the source term in the right-hand side vanishes in the background, we can readily derive the background solution for the differential equation:

$$\bar{\mathcal{D}}_A = -a\lambda = a\bar{r}_\lambda \ . \tag{4.31}$$

With the background solution in the source term, we can integrate the equation twice to obtain the linear-order solution as

$$\left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)^{(1)}_\lambda = \left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)^{(1)}_o + \frac{1}{2}\int_0^\lambda d\lambda'\ \lambda'\ \hat{\mathfrak{R}}_{\lambda'} \ , \tag{4.32}$$

$$\left(\frac{\mathcal{D}_A}{a}\right)^{(1)}_s = \lambda_s\left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)^{(1)}_o + \frac{1}{2}\int_0^{\lambda_s} d\lambda\ (\lambda_s-\lambda)\lambda\ \hat{\mathfrak{R}}_\lambda = \lambda_s\left[\left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)_o - \frac{1}{2}\mathfrak{I}\right] \ , \tag{4.33}$$

where we used $\mathcal{D}_A(\lambda_o) = 0$ and the definition of $\mathfrak{I}$ in Eq. (4.23). Therefore, the angular diameter distance is

$$\mathcal{D}_A = -a_s\lambda_s\left[1 - \left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)^{(1)}_o + \frac{1}{2}\mathfrak{I}\right] = \bar{D}_A(z)\left[1 + \delta z + \frac{\Delta\lambda_s}{\lambda_z} - \left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)^{(1)}_o + \frac{1}{2}\mathfrak{I}\right] \ , \tag{4.34}$$

where we expressed the scale factor a_s and the affine parameter λ_s of the source in terms of its observed redshift and residual perturbations. Comparing to the result in Eq. (4.27), it is clear that the boundary condition for the derivative term at the observer should be

$$\left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)^{(1)}_o = -\widehat{\Delta\nu}_o \ , \tag{4.35}$$

and the solution would be consistent with our previous calculations.

According to [34] (their equation 34), the derivative of the angular diameter distance at the observer is related to the photon energy as

$$\left(\frac{d}{d\Lambda}\mathcal{D}_A\right)_o = -\omega_o \ . \tag{4.36}$$

The derivative with respect to the conformally transformed affine parameter yields

$$\frac{d}{d\lambda}\left(\frac{\mathcal{D}_A}{a}\right) = \mathbb{C}a\frac{d}{d\Lambda}\mathcal{D}_A - H\mathcal{D}_A \ , \qquad \left(\frac{d}{d\lambda}\frac{\mathcal{D}_A}{a}\right)_o = -\left(\mathbb{C}a\omega\right)_o = -\left(1+\widehat{\Delta\nu}\right)_o \ , \tag{4.37}$$

and indeed the boundary condition at the observer is consistent with our expectation.

Despite the derivation here, there exist a few inconsistencies in the results of [11] — they have ignored several terms at the observer position:

$$\delta\eta_o = 0 \ , \qquad \delta\nu_o = 0 \ , \qquad \delta n^i_o = 0 \ . \tag{4.38}$$

The first term $\delta\eta_o$ in Eq. (2.24) cannot be set zero in the conformal Newtonian gauge. Either the second term or the third term may be set zero by using the normalization condition $\mathbb{C}$, but not both at the same time. The absence of these terms breaks the gauge invariance of the luminosity distance. Finally, equation (101) in [11]

$$\frac{1}{\bar{D}_A}\left(\frac{\mathcal{D}_A}{a}\right)^{(1)}_s = -\psi + V_{\parallel_o} + \frac{1}{\bar{r}_z}\int_0^{\bar{r}_z} d\bar{r}\ \left[2\psi - \left(\frac{\bar{r}_z - \bar{r}}{\bar{r}^2}\right)\hat{\nabla}^2\psi\right] \ , \tag{4.39}$$

should correspond in our notation to

$$\frac{1}{\bar{D}_A}\left(\frac{\mathcal{D}_A}{a}\right)^{(1)}_s = \widehat{\Delta\nu}_o + \frac{1}{2}\mathfrak{I} = 2\psi_o - \psi + V_{\parallel_o} - \frac{1}{\bar{r}_z}\int_0^{\bar{r}_z} d\bar{r}\left[\left(\frac{\bar{r}_z - \bar{r}}{\bar{r}}\right)\hat{\nabla}^2\psi + (\bar{r}_z - \bar{r})\, 2\psi'\right] \ , \tag{4.40}$$

where we took $\delta\nu_o = 0$ and $\phi = -\psi$ to facilitate the comparison, as adopted in [11].

5 Jacobi Mapping Approach to the Luminosity Distance

Similar in spirit to the Sachs approach based on the optical scalar equation, another approach to modeling the luminosity distance is to utilize the Jacobi mapping, or the *geodesic deviation* equation (see, e.g., [35, 36] for reviews). We first present the propagation equations for the Jacobi field and then derive the luminosity distance by using the Jacobi field.

5.1 Geodesic Deviation Equation and Jacobi Matrix

We derive how two neighboring light rays from the same source propagate and how they are related to the boundary condition. Consider two light rays at the same affine parameter Λ, separated by an infinitesimal distance $\delta x^a(\Lambda) = \xi^a\delta\Sigma$, where the connecting vector ξ^a is called a Jacobi field and Σ parametrizes the separation δx^a of two rays along the same affine parameter Λ. We will derive the Jacobi field along the main ray, working in the limit $\delta\Sigma \to 0$. Since the Jacobi field connects two rays at the same affine parameter or the same phase ϑ of the wave propagation, it is orthogonal to the photon wavevector, and the Jacobi field can be written in general as

$$\xi^a k_a = 0 \ , \qquad\qquad \xi^a \equiv \xi^I[e_I]^a + \xi^0 k^a \ , \qquad\qquad \mathcal{H}^{ab} = [e_I]^a[e_J]^b\delta^{IJ} \ , \tag{5.1}$$

where we introduced orthonormal tetrads $[e_I]^a$ with $I = 1, 2$ that specify the two-dimensional hypersurface (characterized by the projection tensor $\mathcal{H}_{ab}$) orthogonal to the photon propagation direction. Four tetrads (one time-like $[e_t]^a = u^a$ and three space-like $[e_i]^a$) in Section 2 form a local orthonormal basis in the rest frame of the observer described by u^a. Here two space-like tetrads $[e_I]^a$ are constructed among the three space-like tetrads, but orthogonal to the photon propagation direction N^a in the rest frame. Furthermore, while the nonvanishing component ξ^0 is consistent with the orthogonality condition of the Jacobi field, it bears no physical relevance to quantities of our interest, and we set $\xi^0 = 0$.[5]

The propagation equation for the Jacobi field (or the geodesic deviation equation) can be derived in a way similar to the derivation of Eq. (4.5) (e.g., [35, 36]):

$$\frac{D}{d\Lambda}\xi^a = k^b\nabla_b\xi^a = \xi^b\nabla_b k^a \ , \qquad\qquad \frac{D^2}{d\Lambda^2}\xi^a = k^c\nabla_c(k^b\nabla_b\xi^a) = -R^a_{bcd}k^b\xi^c k^d \ , \tag{5.2}$$

[5] This is in fact possible by constructing a projected Jacobi field $\xi^a_\perp = \mathcal{H}^a_b\xi^b$. However, since we will exclusively work on the projected field, we simply set $\xi^0 = 0$ and call ξ^a a (projected) Jacobi field.

where we have used the symmetry of ξ^a along the geodesic and the geodesic equation for k^a. By transporting the local tetrads along the geodesic, the propagation equations can be converted into the those for the components of the Jacobi field:

$$\frac{d}{d\Lambda}\xi^I = \mathfrak{B}^I_J\xi^J \ , \qquad \frac{d^2}{d\Lambda^2}\xi^I = -\mathfrak{R}^I_J\xi^J \ , \tag{5.3}$$

where we have defined the projected tensors

$$\mathfrak{B}^I_J = (\nabla_b k^a)[e^I]_a[e_J]^b \ , \qquad \mathfrak{R}^I_J = (R^a_{bcd}k^bk^d)[e^I]_a[e_J]^c \ . \tag{5.4}$$

In relation to the propagation equation, the Jacobi matrix $\mathfrak{D}$ simply connects the Jacobi field at one position to another position at Λ:

$$\xi^I(\Lambda) \equiv \mathfrak{D}^I_J(\Lambda)\dot{\xi}^J_o \ , \qquad \dot{\xi}^I_o \equiv \frac{d}{d\Lambda}\xi^I\bigg|_{\Lambda_o} \ , \tag{5.5}$$

where we set the boundary condition at the observer, as the light rays converge at the observer position $\xi^I(\Lambda_o) = 0$. In terms of the Jacobi matrix, the propagation equations can be readily expressed as

$$\frac{d}{d\Lambda}\mathfrak{D}^I_J = \mathfrak{B}^I_K\mathfrak{D}^K_J \ , \qquad \frac{d^2}{d\Lambda^2}\mathfrak{D}^I_J = -\mathfrak{R}^I_K\mathfrak{D}^K_J \ , \tag{5.6}$$

where the boundary condition of the Jacobi field translates into the boundary condition for the Jacobi matrix:

$$\mathfrak{D}^I_J(\Lambda_o) = 0 \ , \qquad \frac{d}{d\Lambda}\mathfrak{D}^I_J\bigg|_{\Lambda_o} = \delta^I_J \ . \tag{5.7}$$

As we derived in Sec. 4.3, the non-vanishing boundary condition $\dot{\xi}^I_o$ can be obtained by considering the Jacobi field near the observer position $\Lambda = \epsilon$, in which

$$\xi^I(\epsilon) = |\Delta t|n^I \ , \qquad \Delta t = \omega_o\epsilon \ , \qquad \dot{\xi}^I_o = -\omega_o n^I \ , \tag{5.8}$$

where Δt is again the infinitesimal time (or distance with $c = 1$) corresponding to the propagation of light from the origin to the physical affine parameter $\Lambda = \epsilon$ and $n^I = (\theta, \phi)$ is the observed angle in a spherical coordinate. Since the physical area at Λ is simply

$$d\mathcal{A}(\Lambda) = \xi^1\xi^2\bigg|_\Lambda = \det\ \mathfrak{D}(\Lambda)\ \dot{\xi}^1_o\dot{\xi}^2_o \ , \tag{5.9}$$

we related the angular diameter distance to the Jacobi map as

$$\mathcal{D}^2_A(\Lambda) = \det\ \mathfrak{D}(\Lambda)\omega^2_o \ . \tag{5.10}$$

5.2 Jacobi Matrix in a Conformal Transformed Metric

As evident in Sec. 4.2, the conformal transformation in Eq. (2.2) gives rise to non-trivial relations to the derivative operators and its associated products. Furthermore, a geodesic path is *no* longer geodesic once conformally transformed, except when it is a null geodesic. Consequently, the propagation equations for the Jacobi matrix need to be carefully transformed, according to the change in the derivative operators.

In a similar way the four velocity u^a transforms, we define conformally transformed tetrads $[\hat{e}_I]^a$ as

$$[\hat{e}_t]^a = \hat{u}^a = au^a \ , \qquad [\hat{e}_I]^a \equiv a[e_I]^a \ , \tag{5.11}$$

and concordantly the Jacobi field in this basis is

$$\xi^a = \xi^I [e_I]^a = \left(\frac{\xi^I}{a}\right) [\hat{e}_I]^a \equiv \hat{\xi}^I [\hat{e}_I]^a \ . \tag{5.12}$$

While there is *no* unique definition for the conformally transformed Jacobi field (e.g., $\hat{\xi}^a$), our definition of $\hat{\xi}^I = \xi^I/a$ is the only choice, with which the propagation equations take the same form in the conformally transformed metric. We will refer $\hat{\xi}^I$ to the conformally transformed Jacobi field.

First, we consider the propagation of the Jacobi field with respect to the conformally transformed affine parameter:

$$\frac{D}{d\lambda}\xi^a = \frac{d\Lambda}{d\lambda}\frac{D}{d\Lambda}\xi^a = \mathbb{C}a^2 \ \xi^b \nabla_b k^a = \xi^b \hat{\nabla}_b \hat{k}^a - \mathcal{H}\xi^0 \hat{k}^a + \xi^a \frac{d}{d\lambda} \ln a \ , \tag{5.13}$$

where we used the relation in Eq. (4.11) and the derivative of the scale factor is

$$\frac{d}{d\lambda}a = \frac{da}{d\eta}\frac{d\eta}{d\lambda} = a'\hat{k}^0 = a\mathcal{H}\hat{k}^0 \ . \tag{5.14}$$

Absorbing the derivative term to the LHS, the propagation of the Jacobi field in the conformally transformed metric is then

$$\frac{D}{d\lambda}\left(\frac{\xi^a}{a}\right) = \left(\frac{\xi^b}{a}\right)\hat{\nabla}_b \hat{k}^a - \mathcal{H}\left(\frac{\xi^0}{a}\right)\hat{k}^a \ , \tag{5.15}$$

and the presence of the second term in contrast to Eq. (5.2) highlights the difference associated with the conformal transformation. Now, we decompose the Jacobi field in terms of the conformally transformed tetrads $[\hat{e}_I]^a$:

$$\text{LHS} = \frac{[\hat{e}_I]^a}{a}\frac{d}{d\lambda}\hat{\xi}^I + \frac{\hat{\xi}^I}{a}\mathcal{H}\hat{k}^0[\hat{e}_I]^a - \frac{\hat{\xi}^I[\hat{e}_I]^a}{a}\frac{d}{d\lambda}\ln a \ , \qquad \text{RHS} = \left(\frac{\hat{\xi}^I[\hat{e}_I]^b}{a}\right)\hat{\nabla}_b\hat{k}^a - \mathcal{H}\left(\frac{\hat{\xi}^I[\hat{e}_I]^0}{a}\right)\hat{k}^a \ , \tag{5.16}$$

where the conformally transformed tetrads obey

$$\frac{D}{d\Lambda}[e_I]^a = \frac{D}{d\Lambda}[e_I]_a = 0 \ , \qquad \frac{D}{d\lambda}[\hat{e}_I]^a = \mathcal{H}\hat{k}^0[\hat{e}_I]^a \ , \qquad \frac{D}{d\lambda}[\hat{e}_I]_a = -\mathcal{H}\hat{k}^0[\hat{e}_I]_a \ . \tag{5.17}$$

Note that the conformally transformed tetrads are *no* longer parallelly transported along the photon path. Multiplying the tetrads on both sides and renaming the indicies, we arrive at

$$\frac{d}{d\lambda}\hat{\xi}^I = (\hat{\nabla}_b\hat{k}^a)[\hat{e}_J]^b[\hat{e}^I]_a\hat{\xi}^J \equiv \hat{\mathfrak{B}}^I_J\hat{\xi}^J \ , \tag{5.18}$$

where the projected tensor $\hat{\mathfrak{B}}^I_J$ takes the same form as in the original definition, but with the conformally transformed metric. This justifies our choice of the definition $\hat{\xi}^I$. Taking another derivative, we complete the derivation of the remaining propagation equation:

$$\begin{aligned}\frac{d^2}{d\lambda^2}\hat{\xi}^I &= -\hat{k}^a{}_{;c}\hat{k}^c{}_{;b}[\hat{e}_J]^b[\hat{e}^I]_a\hat{\xi}^J + \hat{R}^a_{dcb}\hat{k}^c\hat{k}^d[\hat{e}_J]^b[\hat{e}^I]_a\hat{\xi}^J + (\hat{\nabla}_b\hat{k}^a)\hat{\mathfrak{B}}^I_J\frac{d}{d\lambda}\hat{\xi}^J \\ &= -\hat{R}^a_{dbc}\hat{k}^c\hat{k}^d[\hat{e}_J]^b[\hat{e}^I]_a\hat{\xi}^J = -\hat{\mathfrak{R}}^I_J\hat{\xi}^J \ ,\end{aligned} \tag{5.19}$$

where the derivative terms of the tetrads are cancelled.

In relation to the propagation equations in the conformally transformed metric, the conformally transformed Jacobi matrix $\hat{\mathfrak{D}}$ satisfies the same form of the propagation equations

$$\hat{\xi}^I(\lambda) = \hat{\mathfrak{D}}^I_J(\lambda)\dot{\hat{\xi}}^J_o \ , \qquad \frac{d}{d\lambda}\hat{\mathfrak{D}}^I_J = \hat{\mathfrak{B}}^I_K\hat{\mathfrak{D}}^K_J \ , \qquad \frac{d^2}{d\lambda^2}\hat{\mathfrak{D}}^I_J = -\hat{\mathfrak{R}}^I_K\hat{\mathfrak{D}}^K_J \ , \tag{5.20}$$

with the same boundary conditions for $\hat{\mathfrak{D}}^I_J$. However, the boundary condition for the conformally transformed Jacobi field $\hat{\xi}^I$ is different:

$$\hat{\xi}^I(\lambda_o) = 0 \, , \qquad \dot{\hat{\xi}}^I_o \equiv \frac{d}{d\lambda}\hat{\xi}^I\bigg|_{\lambda_o} = -\left(\mathbb{C}a\omega\right)_o n^I = -(1+\widehat{\Delta\nu}_o)n^I \, , \tag{5.21}$$

where the normalization in the background is set $\overline{\mathbb{C}a\omega} = 1$. Therefore, the angular diameter distance is then

$$\mathcal{D}^2_A(\lambda) = a^2_\lambda \det\ \hat{\mathfrak{D}}(\lambda)\,(\mathbb{C}a\omega)^2_o = a^2_\lambda(1+\widehat{\Delta\nu}_o)^2 \det\ \hat{\mathfrak{D}}(\lambda) \, . \tag{5.22}$$

5.3 Fluctuation in the Luminosity Distance

The fluctuation in the luminosity distance will be computed by using the relation of the Jacobi map to the angular diameter distance. To compute the Jacobi map, the propagation equations need to be solved given the boundary condition. The source term of the propagation equation is

$$\hat{\mathfrak{R}}^I_J = (\hat{R}^a_{bcd}\hat{k}^b\hat{k}^d)[\hat{e}^I]_a[\hat{e}_J]^c = \frac{1}{2}\delta^I_J\hat{\mathfrak{R}} + (\hat{C}^a_{bcd}\hat{k}^b\hat{k}^d)[\hat{e}^I]_a[\hat{e}_J]^c \, , \tag{5.23}$$

where we defined the conformal (Weyl) tensor

$$\hat{C}^a_{bcd} = \hat{R}^a_{bcd} - \frac{1}{2}\left(\delta^a_c\hat{R}_{bd} + \hat{g}_{bd}\hat{R}^a_c - \hat{g}_{bc}\hat{R}^a_d - \delta^a_d\hat{R}_{bc}\right) + \frac{\hat{R}}{6}\left(\delta^a_c\hat{g}_{bd} - \delta^a_d\hat{g}_{bc}\right) \, , \tag{5.24}$$

and the trace of the source tensor is $\hat{\mathfrak{R}}^I_I \equiv \hat{\mathfrak{R}} = \hat{R}_{ab}\hat{k}^a\hat{k}^b$ as in Eq. (4.22). With the vanishing source tensor in the background, the propagation equation can be trivially integrated to yield

$$\frac{d}{d\lambda}\bar{\hat{\mathfrak{D}}}{}^I_J(\lambda) = \delta^I_J \, , \qquad \bar{\hat{\mathfrak{D}}}{}^I_J(\lambda) = \lambda\ \delta^I_J = -\bar{r}_\lambda\delta^I_J \, . \tag{5.25}$$

At the linear order in perturbation, the source tensor contributes to the propagation equations, and the integration over the background solution gives

$$\frac{d}{d\lambda}\hat{\mathfrak{D}}^{I(1)}_J(\lambda) = -\int_0^\lambda d\lambda'\ \lambda'\ \hat{\mathfrak{R}}^I_J(\lambda') \, , \qquad \hat{\mathfrak{D}}^{I(1)}_J(\lambda_s) = -\int_0^{\lambda_s} d\lambda\ (\lambda_s-\lambda)\lambda\ \hat{\mathfrak{R}}^I_J(\lambda) \, . \tag{5.26}$$

The determinant of the Jacobi map is therefore

$$\det\hat{\mathfrak{D}}(\lambda_s) = \lambda^2_s + \lambda_s\hat{\mathfrak{D}}^I_I(\lambda_s) = \lambda^2_s\left[1 - \int_0^{\lambda_s} d\lambda\left(\frac{\lambda_s-\lambda}{\lambda_s\lambda}\right)\lambda^2\ \hat{\mathfrak{R}}(\lambda)\right] = \lambda^2_s\,(1+\mathfrak{I}) \, , \tag{5.27}$$

where the integral is exactly the integration of the expansion perturbation $\delta\hat{\theta}$ in Eq. (4.23). Putting it altogether and keeping the linear order terms, the angular diameter distance is derived as

$$\mathcal{D}_A = a_s(1+\widehat{\Delta\nu}_o)\ \det{}^{1/2}\hat{\mathfrak{D}}(\lambda_s) = -a_s\lambda_s\left(1+\widehat{\Delta\nu}_o + \frac{1}{2}\mathfrak{I}\right) = \bar{D}_A(z)\left(1+\delta z + \frac{\delta r_z}{\bar{r}_z} - \kappa + \phi\right) \, , \tag{5.28}$$

consistent with the previous derivations.

5.4 Bonvin, Durrer, Gasparini 2006 [3]

The Jacobi mapping approach was first introduced in modeling the luminosity distance [3]. This approach provides a physically simple description of the light propagation measured by an observer along the photon path, and it can be readily generalized to the weak lensing formalism [35–37]. The Jacobi field in [3] is computed by using the Jacobi (four) vector ξ^a in Eq. (5.1), rather than its projected component ξ^I in our approach. Consequently, the Jacobi matrix becomes a 4-by-4 matrix, in which only two-dimensional subspace carries the relevant information. In [3], the determinant $|\hat{J}|$ of the Jacobi matrix in the conformally transformed metric corresponds to something similar, but different from the determinant of our Jacobi matrix:

$$|\hat{J}|^{1/2} = -\lambda_s(1+\widehat{\Delta\nu}_s)\,(1-2\psi+\Delta\alpha) = -\lambda_s\left(1+\widehat{\Delta\nu}_s+\frac{1}{2}\mathfrak{I}\right) \neq \det\hat{\mathfrak{D}}^{1/2} \ , \tag{5.29}$$

where $\Delta\alpha$ is the notation used in [3] and their $\hat{w} = 1+\widehat{\Delta\nu}$. However, when the physical angular diameter distance is computed as

$$\mathcal{D}_A = a_s\,\frac{\hat{w}_o}{\hat{w}_s}|\hat{J}|^{1/2} = -a_s\lambda_s\left(1+\widehat{\Delta\nu}_o+\frac{1}{2}\mathfrak{I}\right) = \bar{D}_A(z)\left(1+\delta z+\frac{\delta r_z}{\bar{r}_z}-\kappa+\phi\right) \ , \tag{5.30}$$

it can reproduce the correct expression consistent with our previous results, if the terms $\delta\eta_o$ neglected throughout their calculations are reinstated. As their main interest lies in computing the angular power spectrum, the absence of the coordinate lapse $\delta\eta_o$ in their expression of the luminosity distance affects nothing; the monopole and the dipole are often *not* part of the power spectrum analysis.

6 Geodesic Light Cone Approach to the Luminosity Distance

The geodesic light cone (GLC) approach to modeling the luminosity distance and other observables in cosmology was first introduced in [9], and it was further extended [12–14, 26] to higher-order calculations of the luminosity distance.

The geodesic light cone coordinates is similar in spirit to our geometric approach, in which the building blocks of theoretical descriptions are the basic observable quantities such as the observed redshift, the observed angular position of sources, the observed flux, and so on. It differs, however, in that the GLC approach incorporates this idea in its coordinate system $x^a = (w,\tau,\tilde{\theta}^I)$, where w describes the phase of past light cones ($w\sim\vartheta$ in Eq. (2.5)), τ is the proper time of observers moving with time-like velocity u^a, and $\tilde{\theta}^I$ with $I=1,2$ describes the observed direction of the light propagation ($\tilde{\theta}^I\sim n^I$). The FRW metric in an inhomogeneous universe is then described by a GLC coordinate as

$$ds^2 = \Upsilon^2dw^2 - 2\Upsilon dwd\tau + \gamma_{IJ}(d\tilde{\theta}^I - U^Idw)(d\tilde{\theta}^J - U^Jdw) \ , \tag{6.1}$$

and the metric components in the background recover the usual FRW components and justify their physical meaning — the proper time $\tau = t$, the scale factor $\Upsilon = a(\eta)$, the phase of past light cones $w = \bar{r}+\bar{\eta}\propto\vartheta$, a spherical coordinate $\gamma_{IJ}d\tilde{\theta}^Id\tilde{\theta}^J = a^2\bar{r}^2d\Omega$ with the observed angles $\tilde{\theta}^I = n^I$, and the auxiliary function $U^I = 0$.

In the presence of inhomogeneities in the Universe, their physical meaning is rather involved and needs clarification in relation to other gauge choice. Transforming the conformal Newtonian gauge $y^a = (\eta,x,y,z)$ to the GLC coordinate $x^a = (w,\tau,\tilde{\theta}^I)$ at each spacetime point,

$$g^{ab}_{\mathrm{GLC}}(x^e) = \frac{\partial x^a}{\partial y^c}\frac{\partial x^b}{\partial y^d}\,g^{cd}(y^e) = -\frac{1-2\psi}{a^2}\frac{\partial x^a}{\partial\eta}\frac{\partial x^b}{\partial\eta} + \frac{1-2\phi}{a^2}\bar{g}^{ij}\partial_ix^a\partial_jx^b \ , \tag{6.2}$$

we derive a series of differential equations for the GLC metric components:

$$(\tau\tau):\ d\tau = a(1+\psi)d\eta \ , \qquad (ww):\ \frac{\partial}{\partial\eta_+}\bar{w} = 1 \ , \qquad \frac{\partial}{\partial\eta_-}w = \frac{1}{2}(\psi-\phi) \ , \tag{6.3}$$

$$(wI):\ \frac{\partial}{\partial\eta_-}\tilde{\theta}^I = \frac{1}{2}\bar{g}^{IJ}\partial_Jw \ , \qquad (IJ):\ \gamma^{IJ} = \frac{1}{a^2}\left[\bar{g}^{IJ}(1-2\phi)+\bar{g}^{IK}\partial_K\tilde{\theta}^J+\bar{g}^{KJ}\partial_K\tilde{\theta}^I\right] \ , \tag{6.4}$$

$$(w\tau):\ \frac{1}{\Upsilon} = \frac{1}{a}\left(1-\psi+\frac{\partial}{\partial\eta_+}w+\frac{\partial}{\partial\eta_-}w-\frac{1}{a}\frac{\partial}{\partial\bar{r}}\tau\right) \equiv \frac{1}{a}(1-\delta\Upsilon) \ , \tag{6.5}$$

where we defined the background lightcone variables

$$\frac{\partial}{\partial\eta_+} = \frac{1}{2}\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\bar{r}}\right) \ , \qquad \frac{\partial}{\partial\eta_-} = \frac{1}{2}\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\bar{r}}\right) = \frac{1}{2}\frac{d}{d\lambda} = -\frac{1}{2}\frac{d}{d\bar{r}} \ . \tag{6.6}$$

Integrating these differential equations, we obtain

$$\tau = \int_0^\eta d\eta' a(1+\psi) \;, \qquad w = \eta_+ - 2\bar{r}\Psi_{\rm av} \;, \qquad \Psi_{\rm av} \equiv \frac{1}{2\bar{r}}\int_0^{\bar{r}} d\bar{r}\,(\psi-\phi) \;, \tag{6.7}$$

$$\tilde{\theta}^I = \theta^I_o + \int_0^{\bar{r}} d\bar{r}'\bar{g}^{IJ}_{\bar{r}'}\int_0^{\bar{r}'} d\bar{r}''\partial_J(\psi-\phi) \;, \qquad \delta\Upsilon = \psi + \int_0^{\bar{r}} d\bar{r}'(\psi-\phi)' - V_\parallel \;, \tag{6.8}$$

and the GLC variable U^I is *not* needed for our present purposes.

The advantage of the GLC coordinates is the simplicity in expression of observable quantities, although they eventually need to be computed in any of conventional choices of gauge conditions. The null vector k^a of a past light cone is specified by the constant phase $\vartheta \sim w$:

$$k^a = g^{ab}\partial_b w = \left(0, -\frac{1}{\Upsilon}, 0, 0\right) \;, \qquad k_a = g_{ab}k^b = (1,0,0,0) \;, \tag{6.9}$$

where the components and indices are for GLC coordinates. Similarly, the four velocity of geodesic flows with proper time τ is

$$u_a = \frac{\partial_a\tau}{\sqrt{\partial\tau\cdot\partial\tau}} = (0,-1,0,0) \;, \qquad u^a = g^{ab}u_b = \left(\frac{1}{\Upsilon},\; 1,\; \frac{U^I}{\Upsilon}\right) \;. \tag{6.10}$$

Consequently, the observed redshift in GLC coordinates is simply the ratio of Υ:

$$1+z = \frac{(k_\mu u^\mu)_s}{(k_\mu u^\mu)_o} = \frac{\Upsilon_o}{\Upsilon_s} = \frac{1}{a_s}\left(1+\mathcal{H}_o\delta\eta_o+\delta\Upsilon_o-\delta\Upsilon_s\right) \;, \tag{6.11}$$

and indeed its expression in terms of the conformal Newtonian gauge variables matches the distortion in the observed redshift

$$\mathcal{H}_o\delta\eta_o+\delta\Upsilon_o-\delta\Upsilon_s = \delta z \;, \tag{6.12}$$

if the first term $\mathcal{H}_o\delta\eta_o$ neglected in [12, 14] is included. Comparing Eq. (2.33), it is evident that $(\delta\Upsilon_o-\delta\Upsilon_s)$ corresponds to $(\widehat{\Delta\nu}_s-\widehat{\Delta\nu}_o)$ in our notation.

Finally, the fluctuation of the luminosity distance is again obtained by computing the angular diameter distance. In GLC coordinates, a unit area perpendicular to the light propagation in the rest frame of geodesic flows is

$$d\mathcal{A} = \mathcal{D}_A^2 d\Omega_o \propto \sqrt{|\gamma|}\; d^2\tilde{\theta} \;, \qquad \gamma = \det\; \gamma_{IJ} \;, \tag{6.13}$$

and the angular diameter distance is therefore

$$\mathcal{D}_A(\Lambda) = \mathcal{D}_A(\epsilon)\left(\frac{|\gamma(\Lambda)|}{|\gamma(\epsilon)|}\right)^{1/4} \;, \tag{6.14}$$

where we used the affine parameters to indicate where the GLC quantities are evaluated.[6] Using Eq. (6.4), the angular determinant at a given spacetime point is

$$\gamma^{-1} = \bar{\gamma}^{-1}\left[1-4\phi+2\int_0^{\bar{r}} d\bar{r}'\left(\frac{\bar{r}-\bar{r}'}{\bar{r}\bar{r}'}\right)\left(\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)(\psi-\phi)\right] \equiv \frac{1-4\phi+4\tilde{J}_2}{a^4r^4\sin^2\theta} \;, \tag{6.17}$$

[6] In [9], the angular diameter distance is defined by lumping together the quantities near the observer ($\Lambda=\epsilon$) as a proportionality constant c:

$$\mathcal{D}_A^2(\Lambda) \equiv c\sqrt{|\gamma(\Lambda)|} \;, \qquad c^{-1} = \sin\tilde{\theta}_o = \sin\theta_o \;, \tag{6.15}$$

and the constant was obtained by taking the limit $\Lambda\to 0$. However, the computation of this limit is incorrect, because the observed photon direction in FRW frame is different from that in the observer rest frame, as shown in Eq. (2.9). From Eq. (6.18), the correct constant is

$$c = \frac{\left(1+\widehat{\Delta\nu}_o-\phi_o\right)^2}{\sin\theta_o} = \frac{\left(1+V_{\parallel_o}\right)^2}{\sin\theta_o} \;. \tag{6.16}$$

Compared to the derivation the luminosity distance in [9], our derivation in Eq. (6.21) with the correct proportionality constant c yields additional terms $\widehat{\Delta\nu}_o-\phi_o = V_{\parallel_o}$ that cancel the extra velocity at the observer position. This correct normalization is also obtained in their later work [38], but with different approach.

and the ratio of two angular determinants is then

$$\frac{|\gamma(\Lambda)|}{|\gamma(\epsilon)|} = \left(\frac{1-4\phi_o+0}{a_o^4 r_o^4 \sin^2\theta_o}\right)\left(\frac{1-4\phi+4\tilde{J}_2}{a_s^4 r_s^4 \sin^2\theta_s}\right)^{-1} = \left(\frac{a_s r_s}{a_o r_o}\right)^4 (1+\phi-\phi_o-J_2)^4 , \tag{6.18}$$

where we have defined $\tilde{J}_2$ and J_2 through the equations following the convention in [26]:

$$J_2 \equiv \tilde{J}_2 + 1 - \sqrt{\frac{\sin\theta_s}{\sin\theta_o}} = \tilde{J}_2 + \frac{1}{2}\tilde{\theta}^{(1)}\cot\theta_o = \frac{1}{2}\int_0^{\bar{r}_s} d\bar{r}\left(\frac{\bar{r}_s-\bar{r}}{\bar{r}_s\bar{r}}\right)\hat{\nabla}^2(\psi-\phi) = -\frac{1}{2}\mathfrak{I}_C . \tag{6.19}$$

Since the angular diameter distance $\mathcal{D}_A(\epsilon)$ near the observer is related to the distance $\Delta r = -(\mathbb{C}a^2\omega)_o\lambda_\epsilon$ that light travels for Δt in Eq. (4.18), the angular diameter distance can be readily computed as

$$\mathcal{D}_A = -(\mathbb{C}a^2\omega)_o\lambda_\epsilon\left(\frac{a_s r_s}{a_o r_o}\right)\left(1+\phi-\phi_o+\frac{1}{2}\mathfrak{I}_C\right) = a_s r_s\left(1+\widehat{\Delta\nu}_o\right)\left(1+\phi-\phi_o-\kappa-V_{\parallel_o}\right) , \tag{6.20}$$

and the fluctuation in the luminosity distance is then

$$\delta\mathcal{D}_L = \delta\mathcal{D}_A = \delta z + \frac{\delta r}{\bar{r}_z} + \phi - \kappa + \left(\widehat{\Delta\nu}_o - \phi_o - V_{\parallel_o}\right) , \tag{6.21}$$

where the comoving distance at origin is taken to be zero: $r_o = -\lambda_\epsilon \to 0$.

Mind the presence of three extra terms in the parenthesis in Eq. (6.21), compared to the previous derivations. In Eq. (6.13), we assumed that the GLC coordinates $\tilde{\theta}^I$ are the observed angles n^I, i.e., $d\Omega_o = d^2\tilde{\theta}$. However, as apparent in Eq. (2.9), the light propagation direction in GLC coordinates is indeed proportional to

$$\tilde{\theta}^I \propto n^I + \delta n^I , \tag{6.22}$$

and therefore, the equality $d\Omega_o = d^2\tilde{\theta}$ holds only if the photon normalization is set at the observer position

$$0 = \delta n^i_o = n^i(\widehat{\Delta\nu}-\phi)_o - V^i_o . \tag{6.23}$$

This condition yields

$$\widehat{\Delta\nu}_o = \phi_o + V_{\parallel_o} , \qquad \delta\nu_o = -(\psi-\phi)_o , \tag{6.24}$$

putting the derivation in GLC coordinates consistent with other approaches.

The GLC attempts to utilize the observed angle in its angular coordinate to describe observable quantities, but the original calculations in [9, 26, 27] neglected the subtle difference in the observed angle in the observer rest frame and that in the FRW frame, in addition to the absence of the coordinate lapse term $\delta\eta_o$. However, the normalization condition for the angular variables was fixed in [38], which brings the expression fully consistent with other approaches. Our normalization condition provides another way to derive the correct expression for the luminosity distance.

7 Discussion

We have computed the luminosity distance by adopting four different approaches in literature (the geometric, the Sachs, the Jacobi mapping, and the geodesic light cone approaches) and presented a unified treatment of the luminosity distance calculation, facilitating the comparison of the different approaches and verifying the sanity of each approach to modeling the luminosity distance. The advantage and disadvantage of each approach is as follows. We started with the geometric approach in [6–8] as a base for our calculations. With the equivalence principle, gravity affects everything in the same way, including all spectrum of the electromagnetic waves (hence achromatic), and the fluctuation in the luminosity distance should therefore be associated with the geometric distortions of the photon path and its flux. In this regard, the geometric approach provides the simplest description and the physically intuitive interpretation of the luminosity distance, in which the change in the luminosity distance arises from the volume distortion (the radial δr and the angular κ) in conjunction with the change δz in the observed redshift.

The Sachs approach is somewhat mysterious in its original form [1], in deriving the angular diameter distance in Eq. (4.14) and the angular diameter distance $\mathcal{D}_A(\epsilon)$ at the origin, which we clarify in relation to the geometric approach. With the clarification in this work, the rules of computing the luminosity distance are straightforward in this approach, and the seemingly different approach taken in [2, 11] is readily incorporated within the same framework. However, it turns out that the calculation using the Sachs approach is rather redundant, in a way numerous terms in Eq. (4.23) are cancelled with other contributions in the final expression. This redundancy may act as an obstacle in deriving the correct expression of the luminosity distance at the second order, to which the expression is rather lengthy and involved (see, e.g., [10]).

The Jacobi mapping approach [3] has a simple physical interpretation of its expression of the luminosity distance in Eqs. (5.9) and (5.10), although they have to be re-arranged in Eq. (5.22) with the conformally transformed metric. As the conformal transformation relates the derivative structures in Eq. (4.11) with two metric tensors in a non-trivial way, the Jacobi field and its propagation equations also transform in an equally non-trivial way. We have identified a properly well-defined Jacobi field $\hat{\xi}^I$ in the conformally transformed metric, with which the form of the propagation equations is preserved as in the original metric. However, the calculation based on the Jacobi mapping results in the expression similar to the expression obtained in the Sachs approach, and hence the redundancy persists in the Jacobi mapping approach.

The geodesic light cone (GLC) approach [9] is somewhat similar in spirit to the geometric approach, in which the observable quantities form the basis. Since it is relatively new and has been mostly computed in the conformal Newtonian (and recently in the synchronous gauge [38]), further work needs to be done to ensure the sanity of the formalism, at least, by checking the gauge invariance of its expressions of the observable quantities. The great strength of the GLC approach is the simplicity of its expressions for observable quantities. However, since the GLC approach is adopted with the full metric, rather than the conformally transformed metric, its calculations are often more complicated than those in other methods.

In summary, all these four methods, if properly exercised, result in the correct and consistent expression of the luminosity distance, providing solid theoretical frameworks. Given the level of consistency with those in the Sachs and the geometric approaches, we believe that all the four methods are on equal footing and can be readily generalized to compute the higher-order corrections in the luminosity distance, although the calculations in the Jacobi mapping and the GLC approaches are performed only with one or two specific gauge conditions. In particular, the second-order calculations are needed to compute the mean of the luminosity distance in an inhomogeneous universe, in which the comparison among other groups is quite difficult. The unified treatment of the luminosity distance in this work can be used to go beyond the linear order, providing a crucial way of checking the robustness of the calculations and ensuring the consistency of the results from different methods.

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